Essential singularities of rigid analytic functions

Put, Marius van der

Published in:
Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen. Series A:Mathematical Sciences

Publication date:
1981

Proceedings A 84 (4), December 21, 1981

Essential singularities of rigid analytic functions

by Marius van der Put

Dept. of Mathematics, Groningen University, the Netherlands

Communicated by Prof. T.A. Springer at the meeting of October 25, 1980

INTRODUCTION

The Picard theorem for a complex analytic function can be formulated as follows:

"Let f be a holomorphic function on $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ with values in $\mathbb{C} - \{0,1\}$ then f can be extended as meromorphic function on

$$\{z \in \mathbb{C} \mid |z| < 1\}\text{''}.$$

A short proof of this statement would be the following: The group

$$\Gamma(2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSl(2,\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2\right\}$$

acts freely as a group of fractional linear transformations on the upper half-space H. The group has 3 parabolic points and the genus of the corresponding algebraic curve is 0. This means that $H/_{\Gamma(2)} \cong \mathbb{C} - \{0,1\}$ and as a consequence $\pi : H \to \mathbb{C} - \{0,1\}$ is the universal covering of $\mathbb{C} - \{0,1\}$.

Let

$$U_1 = \{z \in \mathbb{C} \mid 0 < |z| < 1, \arg(z) = \pi\};$$
$$U_2 = \{z \in \mathbb{C} \mid 0 < |z| < 1, \arg(z) = 0\}$$

$U_1 \cap U_2 = U^+ \cup U^-$ where

$$U^+ = \{z \in U_1 \cap U_2 \mid \operatorname{im}(z) > 0\}$$

and

$$U^- = \{z \in U_1 \cap U_2 \mid \mathrm{im}(z) < 0\}.$$

There are lifts $f_i : U_i \to H$ of f/U_i (i.e. $\pi \circ f_i = f/U_i$ for $i = 1, 2$) such that $f_1(\frac{1}{2}i) = f_2(\frac{1}{2}i)$. So f_1 coincides with f_2 on U^+. There is a unique $\gamma \in \Gamma(2)$ with $f_1 = \gamma \circ f_2$ on U^-.

We divide H by the action of $\langle\gamma\rangle$, the subgroup of $\Gamma(2)$ generated by γ. The result $H' = H/\langle\gamma\rangle$ is analytically isomorphic to one of the following spaces

(a) $\{z \in \mathbb{C} \mid |z| < 1\}$ if $\gamma = \mathrm{id}$.
(b) $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ if γ is parabolic
(c) $\{z \in \mathbb{C} \mid r < |z| < 1\}$ for some $r > 0$ if γ is hyperbolic.

Let $\pi' : H' \to \mathbb{C} - \{0,1\}$ denote the natural map induced by π. From the above it follows that f lifts to a holomorphic map $F : \{z \in \mathbb{C} \mid 0 < |z| < 1\} \to H'$ such that $\pi' \circ F = f$. Since F is bounded, it follows that F (and so also f) extends to $\{z \in \mathbb{C} \mid |z| < 1\}$.

We consider a field K, complete with respect to a non-archimedean valuation. In order to simplify the exposition we suppose that K is algebraically closed. Let $\mathbb{P} = \mathbb{P}^1(K)$ denote the projective line over K. In many situations one has to study holomorphic or meromorphic functions on an open set $\Omega \subset \mathbb{P}$ of the form $\Omega = \mathbb{P} - L$, where L is a compact set. We call L an essential singularity for the meromorphic function f on Ω if f does not extend to a meromorphic function on any $\Omega' = \mathbb{P} - L'$ where L' is a proper closed subset of L.

If L has at least one isolated point then it turns out that $f(\Omega)$ omits at most one value of $\mathbb{P}$. However if L is perfect then $f(\Omega)$ may omit a finite number of values in $\mathbb{P}$ (§ 2, example 1) or $f(\Omega)$ may even omit a compact infinite subset of $\mathbb{P}$ (§ 2, example 2).

The examples are derived from the theory of discontinuous groups over a non-archimedean valued field. In this respect the theory seems quite far from its archimedean analogue. We refer to [1] and [2] for non-archimedean function theory of one variable and for discontinuous groups.

§ 1. POSITIVE RESULTS ON THE VALUES OF HOLOMORPHIC MAPS

A connected affinoid subset X of $\mathbb{P}$ is a subset of the form $X = \mathbb{P} - (B_1 \cup \ldots \cup B_n)$ where $B_1, \ldots, B_n$ are disjoint open disks in $\mathbb{P}$. The $B_1, \ldots, B_n$ are usually called the holes of X; their number is n.

(1.1) PROPOSITION. Let f be a non-constant holomorphic function on a connected affinoid subset X of $\mathbb{P}$. Then $f(X)$ is a connected affinoid subset of $\mathbb{P}$. Moreover the number of holes of $f(X)$ is less than or equal to the number of holes of X.

PROOF. The canonical reduction $\bar{X}$ of X is the maximal ideal space of $\overline{\mathcal{O}(X)}$ ([2] p. 113). According to [2] p. 78, 79 the ring $\overline{\mathcal{O}(X)}$ has the form $\bar{K}[z_1, \ldots, z_n]/I$

where I is an ideal generated by elements $\bar{E}_{i,j} (i \neq j, 1 \leq i, j \leq n)$ of the form

$$\bar{E}_{ij} = z_i z_j + \alpha_{ij} z_i + \beta_{ij} z_j \text{ with } \alpha_{ij}, \beta_{ij} \in \bar{K}.$$

It follows that each component L of $\bar{X}$ is isomorphic to $\mathbb{P}(\bar{K}) - V(L)$ where $V(L)$ is a finite non-empty subset of $\mathbb{P}(\bar{K})$. We construct $\hat{\bar{X}}$, the completion of $\bar{X}$, by completing each component L of $\bar{X}$ to a $\mathbb{P}(\bar{K})$. The total number of "missing" points of $\bar{X}$ (i.e. the points of $\hat{\bar{X}} - \bar{X}$) is equal to $\sum \# V(L) = n =$ the number of holes of X.

The set $Y = f(X)$ is according to [2] p. 110, lemma (2,7), the union of an affinoid set and a finite set. Since X is connected it follows that Y is actually a connected affinoid subset of $\mathbb{P}$.

The surjective map $f : X \to Y$ induces a morphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ which is an isometry with respect to the spectral norms $\| \ \|_{sp}$ on X and Y. We obtain an induced, injective $\overline{f^*} : \overline{\mathcal{O}(Y)} \to \overline{\mathcal{O}(X)}$ and a surjective (since [2] p. 114, lemma (2.9.1)) morphism $\bar{f} : \bar{X} \to \bar{Y}$.

The restriction of $\bar{f}$ to any component $L = \mathbb{P}(\bar{K}) - V(L)$ of $\bar{X}$ extends uniquely to a morphism of $\mathbb{P}(\bar{K}) \to \hat{\bar{Y}}$. So $\bar{f}$ extends to a morphism $\hat{\bar{f}} : \hat{\bar{X}} \to \hat{\bar{Y}}$. The last map is surjective since $\hat{\bar{f}}(\hat{\bar{X}})$ is complete and contains $\bar{Y}$. Hence the number of missing points of $\bar{Y}$ is $\leq n$. This proves the proposition.

We propose now a second proof of the last statement of the proposition. In [1] § 1, (1.8.9) one has established an exact sequence

$$0 \to A(X) \to \mathcal{O}(X)^* \to \mathbb{Z}^{n-1} \to 0$$

in which $\mathcal{O}(X)^*$ is the group of invertible holomorphic functions on X; n is the number of holes of X; $A(X) = \{\lambda(1+h) \mid \lambda \in K^*, h \in \mathcal{O}(X), \|h\|_{sp} < 1\}$.

Let m be the number of holes of Y. The map f induces $f^* : \mathcal{O}(Y)^* \to \mathcal{O}(X)^*$ such that $(f^*)^{-1}(A(X)) = A(Y)$. So we find an injective map $\mathbb{Z}^{m-1} \to \mathbb{Z}^{n-1}$ and we have shown that $m \leq n$.

(1.2) PROPOSITION. Let L be a compact subset of $\mathbb{P}$ and let $\Omega = \mathbb{P} - L$ denote the analytic subspace of $\mathbb{P}$ defined by the family

$$\{F \mid F \text{ affinoid in } \mathbb{P}; F \cap L = \phi\}.$$

For any non-constant holomorphic map $f : \Omega \to \mathbb{P}$ the set $\mathbb{P} - f(\Omega)$ is compact.

PROOF. We consider the subspace Ω' of $\mathbb{P}$ defined by the family of affinoid sets $\{f(X) \mid X \text{ affinoid}; X \cap L = \phi\}$. If Ω' is not of the form $\mathbb{P} - \{\text{a compact set}\}$ then, according to [2] p. 145, (2.5), there exists a non-constant bounded holomorphic function h on Ω'. The holomorphic function $h \circ f$ on Ω is also bounded and must be constant according to the same result. This implies however that f is constant. So the proposition is proved and we have proved slightly more, namely: every affinoid subset, lying in $f(\Omega)$, is the image of an affinoid subset of Ω under the map f.

(1.3) PROPOSITION. (A version of Picard's theorem). Let f be meromorphic function on $\{z \in K \mid R < |z|\}$ which cannot be extended at ∞. Then f omits at most one value.

PROOF. We note that this result must be known. By lack of reference we include two proofs. Suppose that f omits at least one value, then we may take f to be holomorphic on $\{z \in K \mid R < |z|\}$.

(1) FIRST PROOF. We may express f as a convergent Laurent-series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

which has infinitely many $a_n \neq 0$ for $n > 0$.

For $\varrho \in |K^*|, R < \varrho < \infty$, we form $\max |a_n| \varrho^n = \alpha(\varrho)$ and we denote the smallest integer n with $|a_n| \varrho^n = \alpha(\varrho)$ by $n(\varrho)$.

Clearly $\lim_{\varrho \to \infty} n(\varrho) = \lim_{\varrho \to \infty} \alpha(\varrho) = \infty$. We will suppose that $\varrho \gg R$ such that $n(\varrho) > 0$. The set $X_\varrho = f(\{z \in K \mid |z| = \varrho\})$ can have the following form:

(a) Suppose that there is only one n with $|a_n| \varrho^n = \alpha(\varrho)$, then

$$X_\varrho = \{z \in K \mid |z| = \alpha(\varrho)\}$$

(b) Suppose that there are more positive integers n with $|a_n| \varrho^n = \alpha(\varrho)$, then $X_\varrho = \{z \in K \mid |z| \leq \alpha(\varrho)\}$.

The above follows from the well-known statement:

$$\sum_{n=-\infty}^{\infty} b_n z^n \in \mathcal{O}(\{z \in K \mid |z| = 1\})$$

has no zeros if and only if there is precisely one m with $|b_m| = \max_n |b_n|$.

Situation (b) occurs for an infinite sequence $\varrho_1, \varrho_2, \ldots$ with $\lim \varrho_i = \infty$. Hence $f(\{z \in K \mid R < |z|\}) = K$.

(2) SECOND PROOF. Suppose that the holomorphic map f omits at least two values in $\mathbb{P}$. Then we may suppose that f omits 0 and ∞. In other words $f \in \mathcal{O}(\{z \in K \mid R < |z|\})^*$. Using [1] § 1, (1.8.9) one sees that f has the form $\lambda z^n(1+h)$ where $\lambda \in K^*$, $n \in \mathbb{Z}$ and h is holomorphic on $\{z \in K \mid R < |z|\}$ such that $|h(z)| < 1$ for all z. But then h can be extended to ∞ and so also f extends at ∞.

§ 2. TWO EXAMPLES

(2.1) *The first example* imitates the proof of Picard's theorem that we have given in the introduction.

Let $k = \mathbb{F}_q((1/t))$ be the Laurent-series field in the variable $1/t$ and with coefficients in the finite field $\mathbb{F}_q$. Let K denote the completion of the algebraic closure of k.

The group $\Gamma(t)$ is the subgroup of $\Gamma(1) = PSl(2, \mathbb{F}_q[t])$ consisting of the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ modulo(t).}$$

In [2], Chapter 10, it is calculated that: $\Gamma(t)$ has $(q+1)$ inequivalent parabolic points and that the genus of the corresponding algebraic curve is zero.

So the holomorphic map

$$f: \mathbb{P}(K) - \mathbb{P}(k) \to \mathbb{P}(K) - \mathbb{P}(k)/\Gamma(t) \simeq \mathbb{P}(K) - \mathbb{P}(\mathbb{F}_q)$$

omits exactly $q+1$ values. We still have to verify that f has an essential singularity at the compact subset $\mathbb{P}(k)$ of $\mathbb{P}$.

Let L be the smallest compact subset of $\mathbb{P}$ such that f admits an extension as meromorphic function on $\mathbb{P} - L$. One easily sees that L always exists and that L is invariant under $\Gamma(t)$. If $L \neq \phi$ then L turns out to be $\mathbb{P}(k)$ since it is invariant. Further $L = \phi$ would mean that f is a rational function on $\mathbb{P}$. But only a constant rational function can be invariant under $\Gamma(t)$.

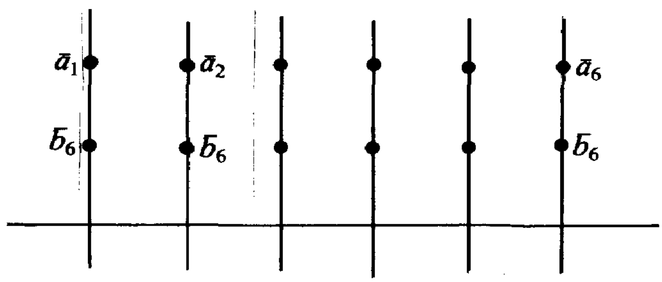
In this example one can clearly vary the finite field $\mathbb{F}_q$ and moreover one can compose f with a rational function on $\mathbb{P}$. This shows the following statement:

"Let the field K have characteristic $\neq 0$ and let $\{a_1, \ldots, a_n\}$ be a subset of $\mathbb{P}(K)$. There exists a perfect compact subset L of $\mathbb{P}(K)$ and a meromorphic function f with an essential singularity at L such that $f(\mathbb{P} - L) = \mathbb{P} - \{a_1, \ldots, a_n\}$".

(2.2) *The second example* works for fields K of any characteristic and residue characteristic. However to simplify matters we assume that the residue field $\bar{K}$ has a characteristic $\neq 2$.

Our construction is a variant of the construction of Whittaker groups done in [2], Chapter 9.

Let the 12 points $a_1, b_1, \ldots, a_6, b_6$ in $\mathbb{P}$ be such that the reduction $\mathbb{P}$ with respect to this set is:



In other terms this means that the position of the 12 points (after an automorphism of $\mathbb{P}$) is such that:

1) all $|a_i| = |b_i| = 1$
2) $|a_i - a_j| = 1$ for $i \neq j$
3) $|b_i - b_j| = 1$ for $i \neq j$
4) $|a_i - b_j| = 1$ for $i \neq j$
5) $|a_i - b_i| < 1$ for all i.

Let s_i $(i = 1, \ldots, 6)$ denote the elliptic element of order 2 with fixed points a_i, b_i. In [2] p. 281 it is shown that the group $\Gamma_0 = \langle s_1, \ldots s_6 \rangle$ generated by the six reflexions is discontinuous and it is shown that the only relations among the generators are $s_1^2 = s_2^2 \ldots = s_6^2 = 1$. Let Ω denote the set of ordinary points of Γ_0.

We introduce now four subgroups Γ_i $(i=1,2,3,4)$ of Γ_0 of finite index. Consider the surjective group homomorphism $\phi : \Gamma_0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ given by $\phi(s_i)=(1,0)$ for $i=1,2,3$ and $\phi(s_i)=(0,1)$ for $i=4,5,6$. The kernel Γ_4 of ϕ is a Schottky group on 9 free generators. The generators are

$$s_1s_2,\ s_1s_3,\ s_4s_5,\ s_4s_6,\ s_1s_4s_1s_2s_4s_1,\ s_1s_4s_1s_3s_4s_1,\ s_1s_5s_4s_1,\ s_1s_6s_4s_1,\ s_1s_4s_1s_4$$

as one easily verifies.

The group Γ_1 is generated by Γ_4 and s_1; the group Γ_2 is generated by Γ_4 and s_4, the group Γ_3 is generated by Γ_4 and s_1s_4. Hence $\Gamma_4 \subset \Gamma_i \subset \Gamma_0$ for $i=1,2,3$ and $[\Gamma_0 : \Gamma_i]=2$ for $i=1,2,3$.

The group Γ_3 turns out to be a free group on 5 generators, namely on $\{s_1s_2, s_1s_3, s_4s_5, s_4s_6, s_1s_4\}$.

The groups Γ_i $(i=0,1,2)$ are not free. One easily calculates that the rank of the abelianized groups $\Gamma_{i/[\Gamma_i,\Gamma_i]}$ is 2 for $i=1,2$.

We write X_i for the algebraic curve Ω/Γ_i $(i=0,\dots,4)$. Although the curve is not always parametrized by a Schottky group (cases $i=0,1,2$) the curve is certainly "locally isomorphic to $\mathbb{P}$" and hence a Mumford curve. (See [2] p. 177). Let g_i denote the genus of X_i, then we have $g_0=0$, $g_1=g_2=2$, $g_3=5$, $g_4=9$ by using [2] p. 250, 251. Moreover we have a diagram of holomorphic maps of degree two between the various curves:

$$\begin{tikzcd}
& X_1 \arrow[dr] & \\
X_4 \arrow[ur] \arrow[r] \arrow[dr] & X_2 \arrow[r] & X_0 \simeq \mathbb{P} \\
& X_3 \arrow[ur] &
\end{tikzcd}$$

We are especially interested in the morphism $X_4 \to X_1$. The curve X_1 is a Mumford curve of genus 2 and can also be parametrized by a Schottky group Δ with Ω' as set of ordinary points.

The map $f : X_4 \to X_1$ lifts to a holomorphic map $F : \Omega \to \Omega'$ since $\pi : \Omega \to X_4$ and $\pi_1 : \Omega' \to X_1$ are the universal coverings. (Compare [2] p. 149–153). The holomorphic map F omits an infinite compact set since $\mathbb{P}-\Omega'$ is infinite.

Our example is completed with the following lemma.

LEMMA. *F has an essential singularity at the compact perfect set $\mathbb{P}-\Omega$.*

PROOF. Using the Riemann-Hurwitz formula one finds that $f : X_4 \to X_1$ is ramified in 12 points. Let $p \in \Omega$ be a point such that its image in X_4 is one of those 12 points. Since $\pi_4 : \Omega \to X_4$ and $\pi_1 : \Omega' \to X_1$ are locally isomorphisms it follows that also F is ramified (of index 2) at p. The whole orbit $\Gamma_4(p)$ consists clearly of ramification points of F. Since p is an ordinary point for Γ_4 the limit points for this orbit are precisely $\mathbb{P}-\Omega$. This implies that F cannot be extended since in any neighbourhood of any $\lambda \in \mathbb{P}-\Omega$ there are infinitely many ramification points of F. So F has an essential singularity at $\mathbb{P}-\Omega$.

REFERENCES

1. Fresnel J. and M. van der Put – Géométrie analytique rigide et applications. Progres in Math. Birkhäuser Verlag '81. (forthcoming).
2. Gerritzen, L. and M. van der Put – Schottky groups and Mumford curves. Lect. Notes in Math. 1980, no. 817.